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FRACTAL CRACKS†

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An attempt is made to take into account the irregular structure of the surface of a real crack in describing the fracture process. The crack surface is modelled using a fractal set of fractional dimension. Using the self-similarity of the fractal, a hierarchical process of the transfer of elastic energy generated during the motion of the crack tip from one scale to another is suggested. Analysis of this process makes it possible to obtain asymptotic expressions for describing the behaviour of the cracks and displacements near the crack tip. It is shown that the fractal geometry of the crack leads to a change in the singular behaviour of the stress fields at the crack tip, and to the appearance of an anomalous dimensionally dependent factor in the expression for the stress intensity factor. Similar results are also obtained for branching fractal cracks. The propagation of a fractal crack in a brittle material is analysed from the positions of the Griffith's criterion.

THE surface of the fracture or crack formed as a result of the failure of most real materials is very irregular and is characterized by the presence of irregularities (peaks, hollows, serrations, etc.) of various different sizes. Therefore, a real crack hardly resembles, within the intermediate scale, ideal cracks with their smooth surfaces, which are usually considered in the theory of fracture. It is clear that the complex structure of the fracture surface which makes a significant contribution to the

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process of crack propagation, should be taken into account when describing the process. During the last few years an important feature of the structure of the crack surface was discovered, namely, the statistical self-similarity of the fracture surface microrelief was established [1, 2].

Taking into account the self-similarity of the crack surface microrelief in modelling its structure leads naturally to the use of fractal surfaces. Models of cracks based on fractal sets were given in [2–6], where experimental methods of determining the fractal dimension of the cracks were also discussed.

When using fractal surfaces (or fractal curves in the plane case) for modelling the cracks, one must remember that from the mathematical point of view the fractal is infinitely tortuous, and the fractal surface (line) has infinite area (length). The tortuousness of the real crack (as well as its area) is naturally finite, and hence a natural lower limit δ of applicability of the fractal model exists. the scale of δ is usually related to the microstructure of the medium. For metals it can be the grain or the subgrain size, for example.

It is also obvious that the applicability of the fractal model must also have an upper limit L. This scale can be related to the geometrical dimensions of the body, to the size of the crack, to the characteristic scale of inhomogeneity of external fields, etc. Thus the fractal model can be used for the intermediate scales l satisfying the condition $\delta \leqslant l \leqslant L$ (we shall leave aside for the time being the problem of the possible multifractal structure of the crack).

The causes and laws of formation of the fractal geometry in the process of fracture have, so far, been insufficiently studied, although there are publications in which models leading to the fractal structure of crack surfaces or of foci of multiple fracture are proposed [6–10]. These problems are not discussed below, and our attention is concentrated on the consequences of the fractal character of the crack surface. It must, however, be remembered that the structure of the fracture surface, its geometrical characteristics and in particular, the fractal dimension D, depend greatly on the mechanism of the fracture process. This is of particular value in interpreting the experimental results, since unlike the many parameters encountered in mechanics of fracture, the fractal dimension of the fracture (crack) D depends not only, and not so much, on the type of material, as on the nature of the fracture process.

1. We shall restrict ourselves to the case of brittle fracture, and first consider the development of cracks in two-dimensional geometry (plane problems). The growth of a crack in a rigid body has, to a considerable extent, a stochastic character and can therefore be modelled by a stochastic fractal. Therefore, the quantities obtained by averaging over the realizations are of the greatest interest in describing the fracture process. Such characteristic quantities are, for example, the asymptotic forms of the displacement and stress fields near the crack tip and the form of functional dependence of the stress intensity factor on the geometrical dimensions of the crack.

If we confine ourselves to the "averaged" characteristics, we can use, as the model of a crack chosen for the analysis, any "typical" realization of the stochastic fractal. Let us consider, as such a realization, a regular geometrical fractal with the corresponding characteristics (dimension, connectedness, etc.).

Thus we shall assume that a crack propagates through the body in question under the action of applied forces, the crack representing a regular geometrical fractal of dimension D (with its proper topological dimension $d = 1$). Figure 1 shows as an example, a fractal crack whose form is that of the classical fractal curve $Kox(D = \ln 4/\ln 3)$.

It is clear that the fractal dimension D falls far short of characterizing the geometry of the crack completely. We shall therefore assume, in addition, that the fractal crack has the form of a twisting line (without branching) distributed "on the average" along the X axis, with the forces applied in the direction of the Y axis. Therefore the fracture takes place, on the macroscale, "on the average" according to type 1 (normal tension crack) and is governed by the stress intensity macrofactor K_I. It is in this sense, of averaging over the possible realizations of the crack trajectory, that the results obtained should be understood.

In the case of a crack with smooth surfaces it is assumed that, when the tip of the crack moves a certain distance Δl, the elastic energy released by this motion is dumped into the crack tip where it

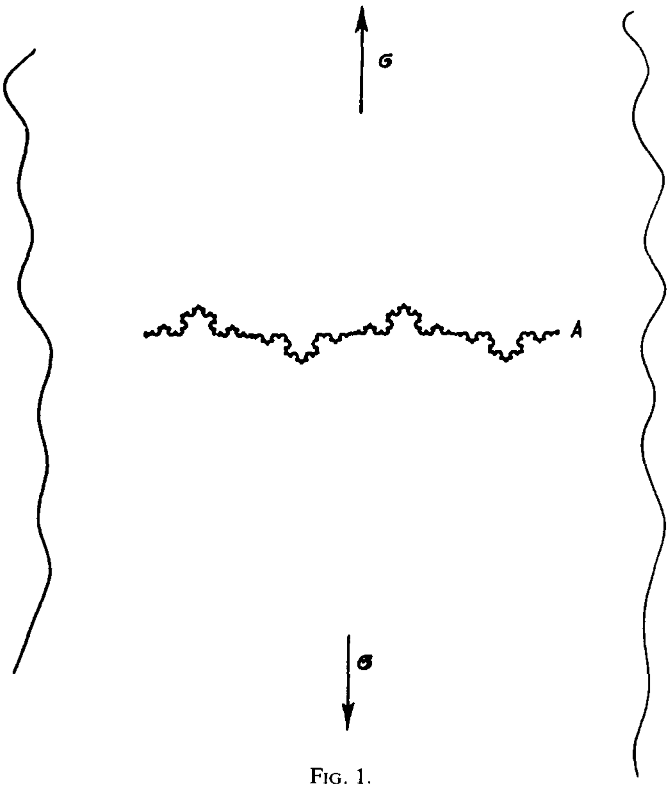

FIG. 1.

dissipates in forming a new crack surface. For a fractal crack the process of elastic energy dissipation is more complicated and interesting, since the fractal has a multiscale hierarchical structure.

Taking into account the hierarchical construction of the fractal crack, we shall assume that the process of fracture (the motion of the crack tip) is accompanied by a cascade transfer of the released elastic energy from the larger scales to the smaller ones, and finally to the microscale where the energy is dissipated and used to form a new surface of discontinuity. We shall assume for the time being that the cascade process occurs with conservation of energy.

Let us describe the process of energy transfer in more detail. First we shall consider the development of the crack on the macroscale $l \gg \delta$, which is nevertheless small compared with the macrodimensions $L\,(l \ll L)$.

When the crack tip moves, on the macroscale l by a distance Δl, the amount of elastic energy released is

$$\Delta U = G_0 \Delta l \tag{1.1}$$

The specific density of released enery $G_0 = \Delta U/\Delta l$ as $\Delta l \to 0$, can be represented in the form of a Γ-integral over the contour γ, representing a circle with centre at the crack tip [11–14]

$$G_0 = \int_\gamma (W n_1 - \sigma_{ij} n_j u_{i,1})\, ds \qquad (i, j = 1, 2) \tag{1.2}$$

Here W is the work done by the stresses on the corresponding deformations (per unit volume), n_i is the outer unit normal to the contour γ, and σ_{ij} and u_i denote the stresses and displacements, respectively.

Since on the macroscale the crack is modelled, on average, by a crack-cut of normal tension, it follows that the elastic field near its ends is characterized by the average stress intensity factor K_I and by the usual asymptotic dependence on the distance r from the corresponding tip of the crack, when the displacements are $\sim r^{1/2}$ and stresses are $\sim r^{-1/2}$.

Let us assume that the following, more general asymptotic relations hold near the tip of the fractal crack:

$$u_i \sim K_I r^{1-\alpha} \varphi_i(\theta), \quad \sigma_{ij} \sim K_I r^{-\alpha} f_{ij}(\theta) \tag{1.3}$$

where K_I is the stress intensity factor, and r is the distance from the tip of the crack. Of course, Eqs (1.3) are assumed to be valid only within the fractal region $\delta \ll r \ll L$. We shall assume that the power index α is, at present, unknown. In the fractal case the index α must depend on the dimensions of the crack.

Substituting relations (1.3) into (1.2) we obtain

$$G_0 \sim E^{-1} K_I^2 l^{1-2\alpha} \tag{1.4}$$

We will now calculate the magnitude of the released elastic energy, by considering the crack on the nth microscale $l_n = l/p^n$ where p is the scale parameter determining the variation in the size of the crack fragments during scaling.

We shall assume that by averaging the crack can also be modelled on the nth microscale using the set of elements of the normal tension cracks. If the crack tip moves, on a macroscale of l, by Δl, on the scale l_n, every fragment of the crack will increase by an amount $\Delta l_n = \Delta l/p^n$. Let B_n be the number of elements of the crack of scale l_n, in which case we can replace relations (1.1) by

$$\Delta U = B_n G_n \Delta l_n$$

where G_n is the density of the elastic energy released on the nth microscale l_n.

Consequently, the cascade process of elastic energy transfer can be described by the following sequences of relations:

$$\Delta U = G_0 \Delta l = B_1 G_1 \Delta l_1 = \ldots = B_n G_n \Delta l_n = \ldots$$

where we have taken into account the fact that the energy released during the passage from the scale l_{n+1} to the scale l_n is conserved. The energy conservation law has the obvious form

$$B_n G_n \Delta l_n = B_{n+1} G_{n+1} \Delta l_{n+1} \tag{1.5}$$

Let us now assume that $G_n = G(l_n)$. Then the energy conservation law (1.5) will enable us to obtain the following renormed group equation:

$$G(l_n) = \frac{B_{n+1}}{B_n} \frac{\Delta l_{n+1}}{\Delta l_n} G\left(\frac{l_{n+1}}{l_n} l_n\right) \tag{1.6}$$

or

$$G(l_n) = \frac{1}{p} \frac{B_{n+1}}{B_n} G\left(\frac{l_n}{p}\right) \tag{1.7}$$

Seeking the solution of equation (1.7) in the form

$$G(l_n) \sim l_n^x \tag{1.8}$$

and making no distinction between the fractal dimension D and self-similarity dimension D_s [1], we obtain

$$x = D - 1 \tag{1.9}$$

In the case of a crack in the form of the curve *Kox* (see Fig. 1) it is clear that $p = 3$, $B_{n+1}/B_n = 4$ and therefore the renormed group equation has the form

$$G(l_n) = {}^4/_3\, G({}^1/_3\, l_n)$$

It can be shown that in this case

$$G(l_n) \sim l_n^{D-1}, \quad D = \ln 4/\ln 3$$

Taking into account the relations (1.8) and (1.9), we have

$$G_0 = G(l) \sim l^{D-1} \tag{1.10}$$

and equating (1.4) and (1.10) we obtain

$$\alpha = (2 - D)/2 \tag{1.11}$$

It follows from this that the fractal geometry of a crack introduces considerable changes to the asymptotic form of elastic fields near its tip.

For a non-fractal crack when $D = 1$ we obtain, as expected, $\alpha = 1/2$. If we model the crack trajectory using the random Brownian process ($D = 1.5$), then $\alpha = 0.25$.

An interesting result is obtained for a fractal crack (more accurately, for a fractal cut) generated by a fractal with a generator shown in Fig. 2(a). It can be confirmed that such a fractal has dimension $D = 2$ and fills the square of mappings shown in Fig. 2(b), densely everywhere. It follows that at the corner A we have a sort of angular notch, but the stresses at this point are singular ($\alpha = 0$). The latter is connected with the fact that the material inside the square, although it is "completely disrupted", is not completely load-free, and this is found to be sufficient for the "suppression" of the singularity at the corner point A.

Similar results are obtained in the case when the crack trajectory on the mesoscale appears to be completely chaotic and can be modelled with the help of a white noise-type random process. As we know, in this case $D = 2$, the stresses near such a crack will be non-singular, and the displacements $\Delta u \sim r$.

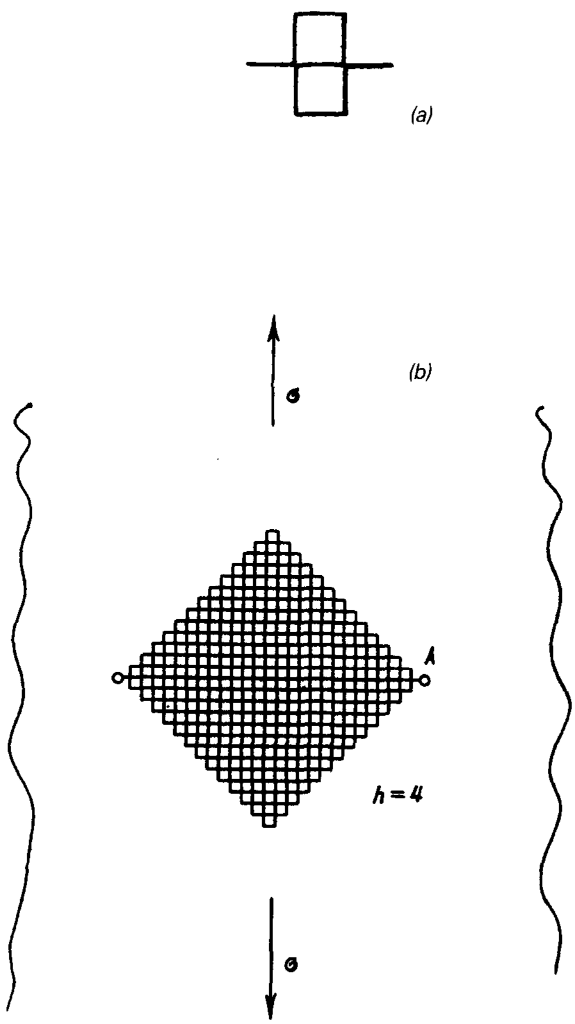

FIG. 2.

Using relations (1.3), (1.11) we obtain the following asymptotic formula for the opening at the tip of the fractal crack:

$$\Delta u \sim r^{D/2}$$

2. Using asymptotic relation (1.3), condition (1.11) and dimensional considerations, we can conclude that the following relation exists between the stress intensity factor K_I and the geometrical size of the crack, for a fractal crack:

$$K_I \sim \sigma l^{\alpha}, \quad \alpha = 1 - D/2 \tag{2.1}$$

The above result can be obtained in a simpler manner based on a Griffith analysis of brittle fracture.

Following Griffith's criterion we shall assume that, in the equilibrium case, an increase in the size of the crack by Δl, the magnitude of the elastic energy released ΔU_e is compensated by an increase in the surface energy ΔU_s along the cut formed, i.e.

$$\Delta U_e = \Delta U_s \tag{2.2}$$

It is clear that

$$\Delta U_e \sim {}^1/_2 E^{-1} \sigma^2 l \Delta l, \quad \Delta U_s = \gamma \Delta s$$

where E is the modulus of elasticity, γ is the coefficient of surface energy, and Δs is the increment in the crack length.

In the fractal case the "true" length of the crack s is connected with its geometric size l by the scaling relations $s \sim (l/\epsilon)^D \epsilon$, where ϵ is the measuring scale. Therefore we have

$$\Delta s \sim (l/\epsilon)^{D-1} \Delta l \tag{2.3}$$

consequently, in the equilibrium case we obtain, from (2.2) and (2.3),

$$\sigma^2 l (l/\epsilon)^{1-D} \sim E\gamma \tag{2.4}$$

Using the definition of the stress intensity factor, we again obtain for the fractal crack a relation of the type (2.1).

Writing $K_I^0 \sim \sigma\sqrt{l}$, we obtain from (2.4) the stress intensity factor for the usual one-dimensional cut of size l, apart from a multiplicative constant of the order of unity

$$K_I^f \sim K_I^0 (l/\epsilon)^{(1-D)/2} \tag{2.5}$$

Thus, when we take into account the fractal geometry of the crack surface, we obtain, in the expression for the stress intensity factor, an unusual scale factor described by the formula (2.5). When D increases (roughly speaking, when the tortuousness of the crack increases), the stress intensity factor decreases and $\ln K_I^f$ will depend linearly on $D-1$. The latter agrees well with experimental results [4].

We note that when $D = 2$, the stress field is non-singular in the neighbourhood of the defect, and that implies that the asymptotic forms of the field σ_{ij} change appreciably, namely the singular terms responsible for the appearance of stress intensity factors vanish from them. In other words, the fractal crack of dimension 2 resembles "on average", a cavity or cavern, and the stress field in the neighbourhood of the defect will be described with help of the "usual" stress concentration factor.

We will transfer the results obtained to the case of cracks in solids. It should merely be remembered that in the three-dimensional case we can consider various different models of cracks with fractal geometry. For example, we can imagine an approximately disc-like crack in the plane, of size R, whose surfaces will be fractals of dimension $2 \leqslant D \leqslant 3$. Using Griffith's criterion we can find that in this case

$$K_I \sim \sigma R^{(3-D)/2}$$

Another fractal model is obtained if we assume the crack surfaces to be smooth, and the crack front to be a fractal curve of dimension $1 \leqslant D \leqslant 2$. For such a model we have

$$K_I \sim \sigma R^{(2-D)/2}$$

Relations (2.4), (2.5) are used in formulating the force criterion of the limit equilibrium of the fractal crack. We shall write it in the form

$$K_I^f = K_{Ic}^f \tag{2.6}$$

where K_{Ic}^f is the critical value of the stress intensity factor characterizing the resistance of the material to the growth of a crack of scale l. Using (2.4) and (2.5) we shall assume that

$$K_{Ic}^f = K_{Ic} l^{(D-1)/2} \tag{2.7}$$

Here K_{Ic} is the resistance of the material to cracks on the macroscale.

We note that in the theory of cracks the quantity K_{Ic} is usually assumed to be a material constant independent of the crack length. Experiments on samples with macrocracks confirm, for a number of materials (especially for brittle and quasi-brittle materials) the validity of this assertion. At the same time, the presence of structural scales in the materials leads to the fact that the magnitude of K_{Ic} may be found to be different for cracks of different scales. The dimension of K_{Ic}, however, is preserved ($[K_{Ic}] = [F]\cdot[L]^{-3/2}$). For a fractal crack the picture is different. When D changes, the parameter K_{Ic}^f changes not only its numerical value, but also the dimension. This implies that the fractal crack resistance of the material is characterized not by a number, but by relation (2.7).

We shall explain how changing over to the usual description of the macrocrack resistance with parameter K_{Ic} can occur in accordance with (2.7). Since $D \geqslant 1$, the fractal crack can propagate in a stable manner as its length and the stress intensity factor increase (naturally, within a specified range). It should be remembered that relation (2.7) holds only on the mesoscale determined by the domain of applicability of the fractal model. When changing from one scale to another (and especially from the mesoscale to the macroscale of the experiments), we must take into account the fact that the fracture mechanisms may also have a characteristic scale of applicability.

The fractal dimension D of the crack depends very much on the mechanism of the fracture. Therefore we must take into account, when changing from one scale to another, the change in dimension D. In other words, we should regard the fracture as a multilevel, multifractal process. The dimension of the fractured structure (in our case, of the crack) will depend on the scale $D = D(l)$. Since the fractal crack appears on the macroscale L simply as a one-dimensional cut, it is natural to assume that $D(L) = 1$.

This, in particular, means that on the macroscale which is usually accessible in the experiments, relation (2.6) becomes saturated and reaches some asymptotic form K_{Ic}^M defining the macrocrack stability of the material. It can be expected that the representations concerning the fractal crack resistance will be found to be useful, especially when analysing short cracks whose controlling laws could not be arrived at from the usual representations of the theory of cracks, in terms of the parameter K_{Ic}. This problem, as well as criterial relations (2.6), (2.7), both need a special experimental study.

The results given above were obtained under the assumption that the energy is conserved in the cascade process. This restriction may be found to be too stringent even in the case of ideally brittle fracture since, as we know, part of the energy is dissipated as a result of dynamic effects at the microlevel. Even more so, when the fracture is not ideally brittle, the energy will be dissipated during the passage from one structural level to another.

The energy dissipation processes during the passage from one structural level to another can easily be taken into account by retaining the assumption on self-modelling (self-similarity) of the transfer process. We shall assume that during the passage from the scale l_{n+1} to the scale l_n, only a qth part of the energy is transferred, and a $(1-q)$th part of the energy is lost, consumed in various dissipative processes accompanying the crack growth. Then the energy conservation law will have to be rewritten, in the cascade process (1.6), as follows:

$$qB_nG_n\Delta l_n = B_{n+1}G_{n+1}\Delta l_{n+1}, \quad q < 1 \tag{2.8}$$

Repeating practically verbatim the above arguments which led to the solution (1.8), we find the energy conservation law in the form (2.7), that

$$G(l_n) \sim l_n^{D-1+\beta}, \quad \beta = -\ln q/\ln p > 0$$

which means that $\alpha = ½(2 - D - \beta)$ and the index β precisely, takes into account the influence of the dissipative processes.

3. Apart from the fractal cracks representing a twisting fractal line, we also have a numerous and important class of cracks which can be successfully described using the methods of fractal geometry. We refer here to branching cracks.

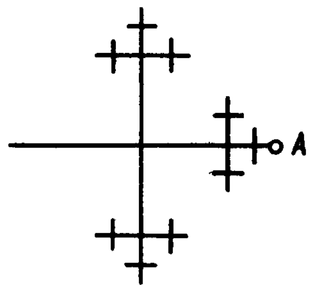

FIG. 3.

Following the arguments and methods used in Sec. 2 for the "twisting" crack, we can also obtain a renormed group equation for the density of the released energy in the case of a self-similar (fractal) branching crack. We shall consider, as an example, the model crack shown in Fig. 3.

In this case equations similar to (1.1)–(1.5) lead to the energy conservation law in the form

$$B_n G_n \Delta l_n = B_{n+1} G_{n+1} \Delta l_{n+1},$$

where G_n is the energy released during the motion of the cracks at the nth level.

The corresponding renormed group equation for $G_n = G(l_n)$ can be written in the form

$$G(l_n) = \frac{B_{n+1}}{B_n} \frac{\Delta l_{n+1}}{\Delta l_n} G\left(\frac{l_{n+1}}{l_n} l_n\right) \tag{3.1}$$

In the case in question (Fig. 3) we have

$$G(l_n) = {}^3/_2 G({}^1/_2 \cdot l_n)$$

For a fractal branching crack, the neighbourhood of the tip of any crack of scale l_n has exactly the same appearance as the neighbourhood of the tip A of the whole crack of the scale l. Therefore, if we assume that $G(l) \sim l^{1-2\alpha}$, then also $G(l_n) \sim l_n^{1-2\alpha}$. Substituting such an expression for G into renormed group equation (3.1), we obtain for α, as previously, expression $\alpha = ½(2-D)$.

An interesting degenerate example of a "branching" crack is given by the Kantor crack shown in Fig. 4. The crack can be regarded as an ensemble of microcracks of size $\Delta \sim ⅓^n$, distributed according to the law of a Kantor set. In this case the singularity at the tip of the "crack" can be calculated directly.

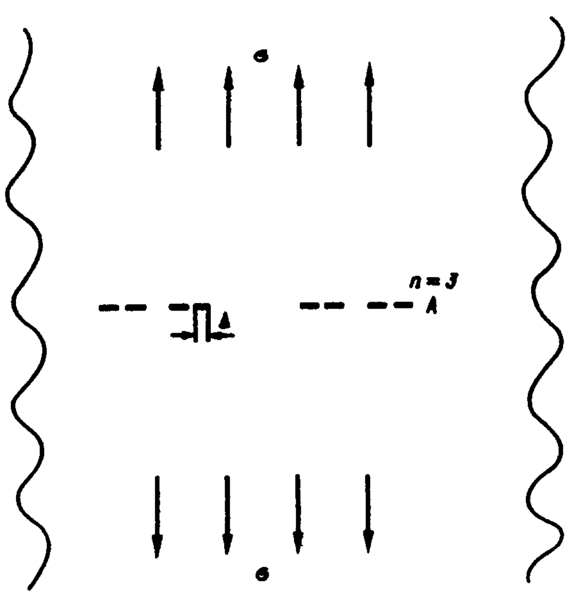

FIG. 4.

Indeed, the quantity G calculated at the macrolevel in the scale l has the form

$$G \sim l^{1-2\alpha} \tag{3.2}$$

On the other hand,

$$G \sim \sum_i G_i \tag{3.3}$$

where G_i is the energy released during the motion of the ith microcrack of scale Δ. Relation (3.3) follows from the fact that the energy is not released (nor dissipated) outside the crack surfaces since the body in question is assumed to be elastic.

Using the elastic field at the crack tip to compute G_i we find $G_i \sim \Delta$. Further, taking into account the fact that $\Delta \sim ⅓^n$, we obtain $G \sim \Sigma_i \Delta \sim 2^n \Delta \sim (⅔)^n$. Putting $l \sim 3^n$ we obtain

$$G \sim l^{D-1}, \quad D = \ln 2/\ln 3 \quad {}^{l}$$

Equating relations (4.2) and (4.4) we see that $\alpha = ½ \cdot (2 - D)$ as expected.

Note. In describing the process of fractal fracture in the propagation of a fractal crack, we can use an approach based on the method of invariant Γ-integrals or Γ-residues [11–14]. It should however be remembered that the Γ-integral usually obtained from (1.2) ceases to be invariant in the fractal case. This is due to the presence of elastic field singularities on all scales under discussion. Taking into account the hierarchical structure of the stress field singularities in the case of fractal cracks, we can write a renormed group equation of the type (3.1) for determining and calculating the invariant Γ-integrals, just as was done for the density of released energy G. In fact, the proposed approach represents one of the methods of "restoring" the invariance of the Γ-integral in the fractal case. The results obtained using this approach are indistinguishable from those given above.

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